The four mirror Sagnac interferometer

Paul B. Andersen

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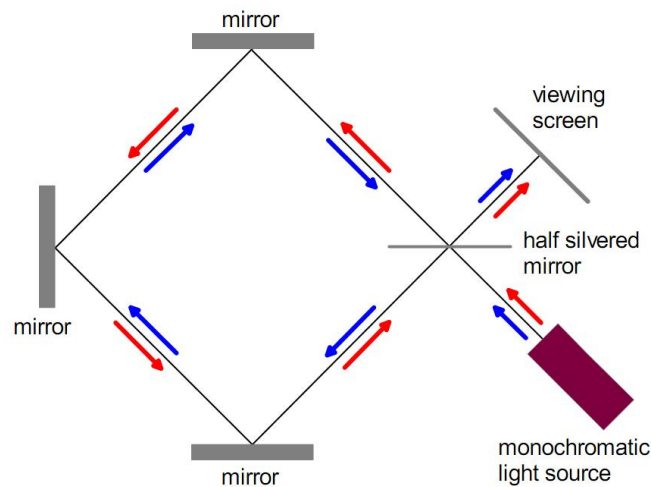## 1 The four mirror Sagnac interferometer

Figure 1: *The four mirror Sagnac interferometer*

Fig.1 shows a four mirror Sagnac interferometer. The light beam from the monochromatic source must be slightly divergent. If a laser is used, it may be necessary to place a diffuser lens in front of it. One part of the beam from the source passes through the half silvered mirror (red arrows) and is reflected off the three other mirrors back to the half silvered mirror. Part of the beam will pass through the half silvered mirror and hit the screen. This beam has gone around the ring in the anticlockwise direction. Another part of the beam from the source will be reflected off the half silvered mirror, and will be reflected around the ring in the clockwise direction (blue arrows). Part of this beam will be reflected off the half silvered mirror and hit the screen. The light from the two beams will form an interference pattern on the screen. The pattern will depend on the collimation of the mirrors, but it will usually be a pattern with bright and dark fringes. If the central parts of the two beams are in phase, there will be a bright fringe at the centre of the screen.

When the Sagnac ring is rotating, the two beams will be slightly deflected in opposite directions. But since the beams are divergent, they will still overlap. The parts of the beams that hit at the centre of the screen will have hit each mirror in the centre. If these parts of the beams are out of phase, the bright fringe will be offset to one side.

We will calculate what the Special Theory of Relativity and emission theories predict the phase difference will be when the interferometer is rotating. By 'emission theory' we will mean a theory where the speed of light in vacuum is c relative to the source, and velocities transform according to the Galilean transform.

2 Calculation of the phase difference between the beams

2.1 Methods

There are two ways of calculating this phase difference:

1. The difference in transit time Δt for the two beams can be calculated.
   The transit time is the time a plane of equal phase uses to move around the ring.
   The phase difference is then $\boldsymbol{\Delta\varphi = 2\pi\nu\,\Delta t}$ where $\boldsymbol{\nu}$ is the frequency of the light.

2. The number of wavelengths in the two beams can be compared.
   The phase difference is then $\boldsymbol{\Delta\varphi = 2\pi\,\Delta N}$ where $\boldsymbol{\Delta N}$ is the difference
   in the number of wavelengths in the two beams.

Even if the two methods necessarily must give the same result, we will calculate the predictions both ways.

2.2 The predictions calculated with the difference in transit times

2.2.1 The geometry of the beams

Given that the centres of the mirrors are a distance $\boldsymbol{r}$ from the centre of the interferometer. The interferometer is rotating in the anticlockwise direction with a peripheral speed $\boldsymbol{v}$. Since all the four parts of the beams between the mirrors will be equal, we can consider only the first part of each beam, as shown in fig. 2.
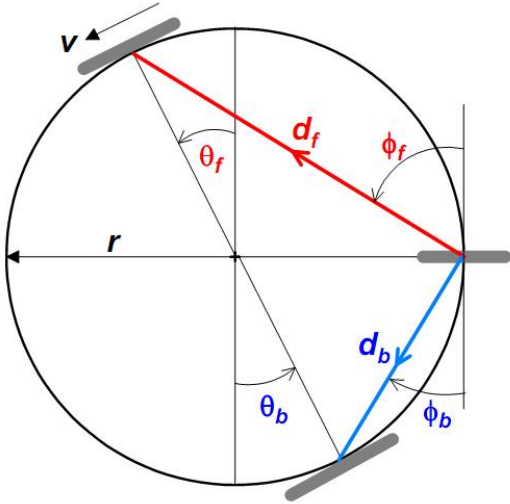

Figure 2: *The geometry of the beams*

Where :

- r is the distance from the centre of the interferometer to the centre of each mirror
- v is the speed of the mirrors in the non rotating inertial frame
- ω is the angular velocity of the mirrors in the non rotating inertial frame, $\omega = v/r$
- ϕ_f is the angle of the central beam emitted in the direction of rotation
- ϕ_b is the angle of the central beam emitted in the direction opposite to the rotation
- d_f is the path length of the central beam emitted in the direction of the rotation
- d_b is the path length of the central beam emitted in the direction opposite to the rotation
- t_f is the transit time of the central beam emitted in the direction of the rotation
- t_b is the transit time of the central beam emitted in the direction opposite to the rotation
- θ_f is the angle a mirror moves during the time t_f
- θ_b is the angle a mirror moves during the time t_b

We have:

$$d_f = \sqrt{(r + r\cos\theta_f)^2 + (r\cos\theta_f)^2} = r\sqrt{2(1 + \sin\theta_f)}$$

$$\theta_f = \omega t_f$$

$$d_f = r\sqrt{2(1 + \sin(\omega t_f))} \tag{1}$$

$$d_b = \sqrt{(r - r\cos\theta_f)^2 + (r\cos\theta_f)^2} = r\sqrt{2(1 - \sin\theta_f)}$$

$$\theta_b = \omega t_b$$

$$d_b = r\sqrt{2(1 - \sin(\omega t_b))} \tag{2}$$

$$\cos\phi_f = \frac{r + r\sin(\omega t_f)}{d_f} = \frac{1 + \sin(\omega t_f)}{\sqrt{2(1 + \sin(\omega t_f))}} = \frac{1}{\sqrt{2}}\sqrt{1 + \sin(\omega t_f)} \tag{3}$$

$$\cos\phi_b = \frac{r - r\sin(\omega t_b)}{d_b} = \frac{1 - \sin(\omega t_b)}{\sqrt{2(1 - \sin(\omega t_f))}} = \frac{1}{\sqrt{2}}\sqrt{1 - \sin(\omega t_b)} \tag{4}$$

Since the Sagnac effect is a first order effect in v/c, first order approximations will be satisfactory.

First order approximations of the above equations will be:

$$d_f \approx r\sqrt{2}\left(1 + \frac{\omega t_f}{2}\right) \tag{5}$$

$$d_b \approx r\sqrt{2}\left(1 - \frac{\omega t_b}{2}\right) \tag{6}$$

$$\cos\phi_f \approx \frac{1}{\sqrt{2}}\left(1 + \frac{\omega t_f}{2}\right) \tag{7}$$

$$\cos\phi_b \approx \frac{1}{\sqrt{2}}\left(1 - \frac{\omega t_b}{2}\right) \tag{8}$$

2.2.2   Prediction of the Special Theory of Relativity

According to this theory, the speed of light is c in the non rotating inertial frame.

For the beam moving with the rotation, we have:

$$c\,t_f = d_f$$

$$ct_f \approx \sqrt{2}\,r \left(1 + \frac{\omega}{2} t_f\right)$$

$$t_f \approx \frac{\sqrt{2}\,r}{c - \frac{\omega r}{\sqrt{2}}} \tag{9}$$

For the beam moving in the opposite direction, we have:

$$ct_b = d_b$$

$$ct_b \approx \sqrt{2}\,r \left(1 - \frac{\omega}{2} t_b\right)$$

$$t_b \approx \frac{\sqrt{2}\,r}{c + \frac{\omega r}{\sqrt{2}}} \tag{10}$$

The difference in transit times will then be:

$$\Delta t = 4\,(t_f - t_r) \approx 4 \left(\frac{\sqrt{2}\,r}{c - \frac{\omega r}{\sqrt{2}}} - \frac{\sqrt{2}\,r}{c + \frac{\omega r}{\sqrt{2}}}\right) = \frac{8\omega r^2}{c^2 \left(1 - \frac{1}{2}\left(\frac{\omega r}{c}\right)^2\right)}$$

Ignoring second order terms in $\omega r/c = v/c$, and inserting the area enclosed by the light beams $A = 2r^2$ yields:

$$\Delta t \approx \frac{4A\omega}{c^2} \tag{11}$$

According to the Special Theory of Relativisty the phase difference between the two beams is:

$$\Delta\varphi = 2\pi\nu\,\Delta t \approx \frac{8\pi A\omega}{\lambda c} \tag{12}$$

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2.2.3 Prediction of emission theories

According to these theories, the speed of light is c relative to the source, and velocities transform according to the Galilean transform. The speed of the two light beams in the non rotating inertial frame will then be:

The beam moving with the rotation:

$$c_f = \sqrt{(c\,\sin\phi_f)^2 + (c\,\cos\phi_f + v)^2} = c\,\sqrt{1 + \frac{2v}{c}\cos\phi_f + \frac{v^2}{c^2}}$$

Using equation (3), and ignoring second order terms yields:

$$c_f \approx c\left(1 + \frac{\omega r}{\sqrt{2}\,c}\right) \tag{13}$$

The beam moving in the opposite direction:

$$c_b = \sqrt{(c\,\sin\phi_f)^2 + (c\,\cos\phi_f - v)^2} = c\,\sqrt{1 - \frac{2v}{c}\cos\phi_b + \frac{v^2}{c^2}}$$

Using equation (4), and ignoring second order terms yields:

$$c_b \approx c\left(1 - \frac{\omega r}{\sqrt{2}\,c}\right) \tag{14}$$

The transit time for the beam going with the rotation:

$$c_f\,t_f = d_f$$

$$c\left(1 + \frac{\omega r}{\sqrt{2}\,c}\right)t_f \approx \sqrt{2}\,r\left(1 + \frac{\omega}{2}t_f\right)$$

$$t_f \approx \frac{\sqrt{2}\,r}{c} \tag{15}$$

The transit time for the beam going in the opposite direction:

$$c_b\,t_b = d_b$$

$$c\left(1 - \frac{\omega r}{\sqrt{2}\,c}\right)t_f \approx \sqrt{2}\,r\left(1 - \frac{\omega}{2}t_f\right)$$

$$t_b \approx \frac{\sqrt{2}\,r}{c} \tag{16}$$

The difference in transit times:

$$\Delta t = 4\,(t_f - t_b) \approx 0 \tag{17}$$

The predicted phase difference is thus:

$$\Delta\varphi \approx 0 \tag{18}$$

According to emission theories, the phase difference has no first order dependency on the rotation.

2.3 The predictions calculated by comparing number of wavelengths

2.3.1 The lengths of the beams

Fig. 3 shows an instant image of the central part of the beams, drawn in the non rotating inertial frame. The beams are slightly curved because the different parts of the beams were emitted at different angles as measured in the non rotating frame. The beam going with the rotation (red curve) is slightly concave, while the beam going in the opposite direction (blue curve) is slightly convex.

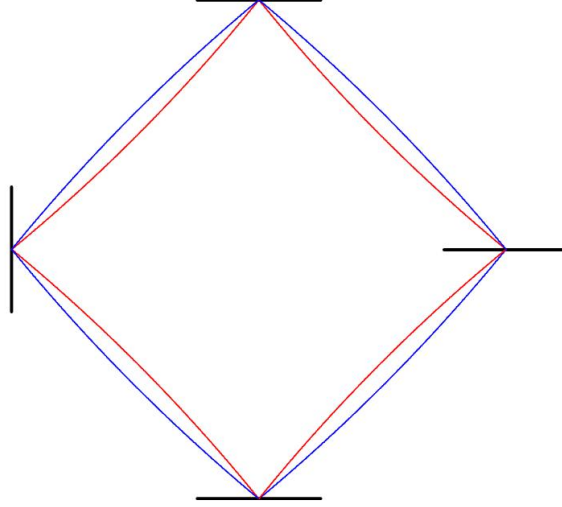We will see if the curvature of the beams has a first order influence on their lengths.

Figure 3: *The shape of the beams as viewed in the non rotating frame*

At the mirrors, the angles to the mirrors are ϕ_f and ϕ_b respectively, as defined in fig. 2.

Combining equation (7) and (9) or (13) gives a first order approximation of $\cos \phi_f$:

$$\cos \phi_f \approx \frac{1}{\sqrt{2}} \left(1 - \frac{\omega r}{\sqrt{2}c} \right) \tag{19}$$

Combining equation (8) and (10) or (14) gives a first order approximation of $\cos \phi_b$:

$$\cos \phi_b \approx \frac{1}{\sqrt{2}} \left(1 + \frac{\omega r}{\sqrt{2}c} \right) \tag{20}$$

(It does not matter if we use the predictions of the Special Theory of Relativity, or emission theories for t_f and t_b; the first order approximations above remain the same.)

Note that equations (19) and (20) show that when $\omega = 0$, $\phi_f = \phi_b = \frac{\pi}{4}$.

A first order Taylor expansion of $\cos\phi$ where $\phi = \frac{\pi}{4} \pm \delta$ yields:

$$\cos\left(\frac{\pi}{4} \pm \delta\right) \approx \frac{1}{\sqrt{2}}(1 \mp \delta) \tag{21}$$

Comparing (19) and (21), and (20) and (21) shows that first order approximations of the angles are:

$$\phi_f \approx \frac{\pi}{4} - \delta \qquad \phi_b \approx \frac{\pi}{4} + \delta \qquad where \; \delta = \frac{\omega r}{\sqrt{2}c}$$

Going back to fig.3, we will see that the curved beams must be sections of circles. We will find the lengths of these sections.
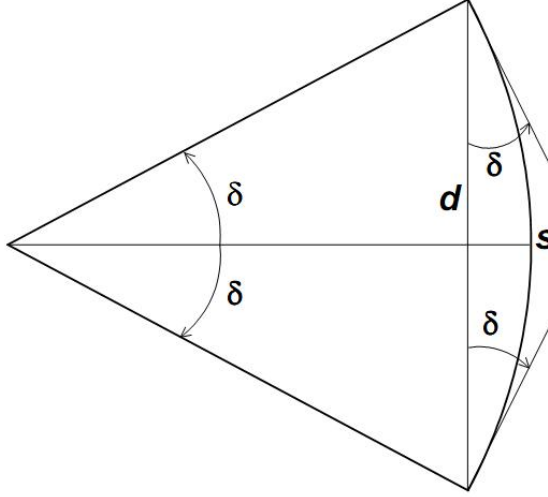

Figure 4: *The length of the curved beam*

From fig. 4, we find: $s = d\,\dfrac{\delta}{\sin\delta}$ where d is the distance between the mirrors, $d = \sqrt{2}r$.

A Taylor expansion of $\sin\delta$ yields $sin\delta \approx \delta - \dfrac{\delta^3}{6}$.

Hence:

$$s \approx d\left(1 + \frac{\delta^2}{6}\right) \approx \sqrt{2}\,r\left(1 + \frac{1}{12}\left(\frac{\omega r}{c}\right)^2\right) \tag{22}$$

This shows that the curvature of the beams will not give a first order addition to their lengths.

We can thus consider the total lengths of both beams to be $4\sqrt{2}\,r$.

2.3.2 Prediction of the Special Theory of Relativity

Since the source is moving in the inertial frame where the centre of the interferometer is stationary, the wavelengths of the light in the beams will according to the Special Theory of Relativity be Doppler shifted.

The wavelength of the light in the beam going with the rotation, measured in the non rotating frame, will be:

$$\lambda_f = \frac{1 - \frac{\omega r}{c}\cos\phi_f}{\sqrt{1 - \left(\frac{\omega r}{c}\right)^2}}\,\lambda \approx \frac{1 - \frac{\omega r}{c}\left(\frac{1}{\sqrt{2}}\left(1 - \frac{\omega r}{\sqrt{2}c}\right)\right)}{\sqrt{1 - \left(\frac{\omega r}{c}\right)^2}}\,\lambda \approx \left(1 - \frac{\omega r}{\sqrt{2}c}\right)\lambda \tag{23}$$

where we have ignored second and higher order terms in $\frac{\omega r}{c}$.

The wavelength of the light in the beam going in the opposite direction, measured in the non rotating frame, will be:

$$\lambda_b = \frac{1 + \frac{\omega r}{c}\cos\phi_b}{\sqrt{1 - \left(\frac{\omega r}{c}\right)^2}}\,\lambda \approx \frac{1 + \frac{\omega r}{c}\left(\frac{1}{\sqrt{2}}\left(1 + \frac{\omega r}{\sqrt{2}c}\right)\right)}{\sqrt{1 - \left(\frac{\omega r}{c}\right)^2}}\,\lambda \approx \left(1 + \frac{\omega r}{\sqrt{2}c}\right)\lambda \tag{24}$$

again ignored second and higher order terms in $\frac{\omega r}{c}$.

The number of wavelengths in the beam going with the rotation becomes:

$$N_f = \frac{4\sqrt{2}\,r}{\lambda_f} \approx \frac{4\sqrt{2}\,r}{\left(1 - \frac{\omega r}{\sqrt{2}c}\right)\lambda} \tag{25}$$

The number of wavelengths in the beam going in the opposite direction becomes:

$$N_b = \frac{4\sqrt{2}\,r}{\lambda_b} \approx \frac{4\sqrt{2}\,r}{\left(1 + \frac{\omega r}{\sqrt{2}c}\right)\lambda} \tag{26}$$

The difference in the number of wavelengths becomes:

$$\Delta N = N_f - N_b \approx \frac{4\sqrt{2}\,r}{\left(1 - \frac{\omega r}{\sqrt{2}c}\right)\lambda} - \frac{4\sqrt{2}\,r}{\left(1 + \frac{\omega r}{\sqrt{2}c}\right)\lambda} = \frac{8\omega r^2}{\lambda c\left(1 - \frac{1}{2}\left(\frac{\omega r}{c}\right)^2\right)} \approx \frac{8\omega r^2}{\lambda c} \tag{27}$$

Inserting the area enclosed by the light beam, $A = 2r^2$ yields:

$$\Delta N \approx \frac{4A\omega}{\lambda c} \tag{28}$$

The predicted phase difference is thus:

$$\Delta\varphi = 2\pi\Delta N \approx \frac{8\pi A\omega}{\lambda c} \tag{29}$$

which is identical to equation (12), as it must be.

2.3.3 Prediction of an emission theory

According to emission theories wavelengths are not Doppler shifted. This is a consequence of the Galilean transform.

The number of wavelengths in the two beams is therefore:

$$N_f = \frac{4\sqrt{2}r}{\lambda} \tag{30}$$

$$N_b = \frac{4\sqrt{2}r}{\lambda} \tag{31}$$

And the difference in the number of wavelengths is:

$$N_f = N_f - N_b = 0 \tag{32}$$

The predicted phase difference is thus:

$$\Delta\varphi = 0 \tag{33}$$

According to the emission theories, the phase difference has no first order dependency on the rotation.

3 Conclusion

We have shown that the Special Theory of Relativity predicts the phase difference between the beams in a rotating four mirror Sagnac ring to be

$$\Delta\varphi \approx \frac{8\pi A\omega}{\lambda c}$$

Since this is in accordance with experimental evidence within the precision of the measurement, the Sagnac experiment confirms the Special Theory of Relativity.

We have shown that emission theories predict that the phase difference between the beams in a rotating four mirror Sagnac ring has no first order dependency on the angular velocity, while experimental evidence show that the dependency is given by the equation above, the Sagnac experiment falsifies emission theories.