A Hafele & Keating like thought experiment

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1 Introduction

This is a calculation of what The General Theory of Relativity predicts for an idealized experiment, similar to the Hafele & Keating experiment.

J.C.Hafele & R.E.Keating: Around-the-World Atomic Clocks

2 The thought experiment

Given three clocks, A, B and C. Clock A is stationary on the ground at equator, while clock B and C are flown in aeroplanes at constant altitude and ground speed in opposite directions around the Earth at equator. At some instant, the clocks are co-located when they all are set to zero. When clock B and C have flown once around the Earth and again are co-located, they are compared to clock A.

3 The proper time of a clock circling the Earth

The Schwarzschild metric is used to find the proper time of clocks in the vicinity of the Earth:

$$c^2 \mathrm{d}\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \mathrm{d}t^2 - \frac{1}{\left(1 - \frac{2GM}{c^2 r}\right)} \mathrm{d}r^2 - r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2\right) \tag{1}$$

where:

$\boldsymbol{\tau}$ is the proper time
$\boldsymbol{t}$ is the Schwarzschild temporal coordinate
$\boldsymbol{r}$ is the Schwarzschild radial coordinate
$\boldsymbol{\theta}$ is the colatitude (angle from north)
$\boldsymbol{\varphi}$ is the longitude
$\boldsymbol{G}$ is the gravitational constant
$\boldsymbol{M}$ is the mass of the Earth
$\boldsymbol{c}$ is the speed of light in vacuum

If the trajectory of the clock is a circle in the equatorial plane, then r and θ are constants, $\theta = \frac{\pi}{2}$ and $r\,\mathrm{d}\varphi = v\,\mathrm{d}t$ where v is the clock's speed measured in the Schwarzschild frame of reference, and equation (1) simplifies to:

$$\mathrm{d}\tau^2 = \left(1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}\right)\mathrm{d}t^2 \tag{2}$$

or:

$$\mathrm{d}\tau = \sqrt{1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}}\,\mathrm{d}t \tag{3}$$

A first order approximation is:

$$\mathrm{d}\tau = \left(1 - \frac{GM}{c^2 r} - \frac{v^2}{2c^2}\right)\mathrm{d}t \tag{4}$$

Assuming r and v are constants and integrating:

$$\tau = \left(1 - \frac{GM}{c^2 r} - \frac{v^2}{2c^2}\right)t + \tau\left(0\right) \tag{5}$$

The Schwarzschild coordinate time t is however a theoretical time shown by no clock, so an equation comparing the proper times of two clocks is more interesting. If we have one clock with proper time τ_1, going with speed v_1 at radial distance r_1, and another clock with proper time τ_2, going with speed v_2 at radial distance r_2, we can write:

$$\mathrm{d}\tau_1 = \left(1 - \frac{GM}{c^2 r_1} - \frac{v_1^2}{2c^2}\right)\mathrm{d}t \tag{6}$$

$$\mathrm{d}\tau_2 = \left(1 - \frac{GM}{c^2 r_2} - \frac{v_2^2}{2c^2}\right)\mathrm{d}t \tag{7}$$

Combining these, we find:

$$\mathrm{d}\tau_2 = \left(\frac{1 - \frac{GM}{c^2 r_2} - \frac{v_2^2}{2c^2}}{1 - \frac{GM}{c^2 r_1} - \frac{v_1^1}{2c^2}}\right)\mathrm{d}\tau_1 \simeq \left(1 + \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{v_1^2 - v_2^2}{2c^2}\right)\mathrm{d}\tau_1 \tag{8}$$

Assuming that $\tau_2 = 0$ when $\tau_1 = 0$ and integrating:

$$\tau_2 \simeq \left(1 + \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{v_1^2 - v_2^2}{2c^2}\right)\tau_1 \tag{9}$$

The difference between the proper times of the clocks will then be:

$$\tau_2 - \tau_1 = \left(\frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{v_1^2 - v_2^2}{2c^2}\right)\tau_1 \tag{10}$$

In the special case when the first clock is at the geoid, r_1 is the radius of the Earth R. If h is the altitude of the second clock, we can write:

$$\tau_2 - \tau_1 = \left(\frac{GM}{c^2}\left(\frac{1}{R} - \frac{1}{R+h}\right) + \frac{v_1^2 - v_2^2}{2c^2}\right)\tau_1 \tag{11}$$

If we assume that $\frac{h}{R} \ll 1$ and insert the gravitational acceleration at the geoid $g = \frac{GM}{R^2}$, the equation can be simplified to:

$$\tau_2 - \tau_1 = \left(\frac{gh}{c^2} + \frac{v_1^2 - v_2^2}{2c^2} \right) \tau_1 \tag{12}$$

4 Calculation of the proper times of the clocks in the thought experiment

From equation (12) we have:

$$\tau_B - \tau_A = \left(\frac{gh_B}{c^2} + \frac{v_A^2 - v_B^2}{2c^2} \right) \tau_A \tag{13}$$

and:

$$\tau_C - \tau_A = \left(\frac{gh_C}{c^2} + \frac{v_A^2 - v_C^2}{2c^2} \right) \tau_A \tag{14}$$

Where τ_A and v_A are respectively the proper time and the speed of the ground clock A, τ_B, v_B and h_B are respectively the proper time, speed and altitude of the west going clock B, and τ_C, v_C and h_C are respectively the proper time, speed and altitude of the east going clock C.

The speeds are here referred to the non-rotating Earth centred frame of reference.

Let's suppose the ground speeds of both the aeroplanes are 232.55 m/s, and their altitude is 9000m. Since the speed of the aeroplanes is half the peripheral velocity of the Earth, they will use two sidereal days on the journey around the Earth, that is τ_A = two sidereal days. These speeds and altitudes are reasonable for commercial aeroplanes, and are close to the speeds and altitudes of the aeroplanes in the Hafele & Keating experiment. We will assume the aeroplanes are flying non stop, though, which obviously was not the case in the H&K experiment.

We can now sum up the data for the clocks:

Speed of ground clock A:	v_A	= 465.1 m/s
Speed of west going clock B:	v_B	= v_A - 232.55 m/s = 232.55 m/s
Speed of east going clock C:	v_C	= v_A + 232.55 m/s = 697.65 m/s
Altitude of clock B:	h_B	= 9000 m
Altitude of clock C:	h_C	= 9000 m
The proper time of clock A:	τ_A	= 172320 seconds
Gravitational acceleration:	g	= 9.800 m/s^2

Inserting these data into equations (13) and (14) yields:
$\tau_B - \tau_A = 325\ ns$, the west going clock gains 325 ns on the ground clock. (H&K 273 ns)
$\tau_C - \tau_A = -90\ ns$, the east going clock loses 90 ns on the ground clock. (H&K 59 ns)